Some Non-Isomorphisms between Pitch and Time
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SOME NON-ISOMORPHISMS BETWEEN PITCH AND TIME

Justin London

1. Claims of Isomorphism

The idea that a single principle governs each and every musical parameter goes back at least to the Pythagoreans, who believed that melody, harmony and rhythm were all ruled by ratio and proportion. Perhaps ever since, western musicians and music theorists have sought universal principles of musical structure. At bottom of such claims is the belief that pitch and time are somehow isomorphic. Faith in the isomorphism of pitch and time has continued through the twentieth-century and to the present. I begin with a few examples.

Isomorphism was an attractive idea to many twentieth-century composers. Messiaen (1956, 13) catalogued symmetrical pitch patterns in his "modes of limited transposition" and found analogous symmetries in his non-retrogradeable rhythms. Similarly, Boulez draws the distinction between "smooth" versus "striated" varieties of space (for pitches) and time (for durations): "Pulsation is for striated time what temperament is for striated space; it has been shown that, depending on whether a partition is fixed or variable, defined space will be regular or irregular; similarly, that the pulsation of striated time will be regular or irregular, but systematic" (1971, 91). Stockhausen (1959) goes much further, for if pitches and rhythms both involve periodic phases between successive impulses, then both are instances of the same basic phenomenon, but in different octaves. Indeed, Stockhausen drew explicit parallels between the overtone series for pitch and categorical values for duration. In the realm of 12-tone compositional theory and method there have been various attempts at translating pitch and pitch-class relationships to the temporal domain—indeed, this is a basic tenet of multi-serialism. Thus, to choose an obvious example, Babbitt (1972) gives a systematic account of how row elements and intervals can (as well as cannot) be related to metric position and relative duration.

Shifting to more recent music theory, Lewin has discussed pitch-time isomorphisms in specific analytical contexts (1981) and as a feature of his well-known models for interval systems (1987).[1] The notions of consonance and dissonance have been applied to metrical relationships, as in Yeston 1976 and (especially) Krebs 1987, 1997. Finally, Pressing (1983) has explicitly claimed that there is a cognitive isomorphism (grounded in musical substance) between the interval pattern of the diatonic scale and certain non-western rhythmic patterns.

At first blush the intuitions of these composers and theorists regarding pitch and time seem quite reasonable. Individual note onsets take specific locations in both measured and unmeasured time, just as individual pitches take discrete locations within the pitch continuum and as scale steps. Similarly, both relative durations and the intervals between pitches can be discussed in terms of ratios or proportions. Pitch classes, beat classes (in an established meter) and scaled durations are all amenable to set and/or group-theoretic treatment. When the same ratios and patterns crop up in both tonal and temporal domains our suspicions are naturally aroused—could the recurrence of a 2–2–1–2–2–2–1 pattern be a sign of some deep parallel between scales and meters?

In a word, no. In the following pages I will present spatial representations—graphs—of pitch and metric systems, starting with familiar tonnetz representations of pitch and pitch-class space. Using mathematical graph theory, I will show that pitch/pitch-class space and an analogous "meter/tempo space" are fundamentally non-isomorphic. This non-isomorphism stems from the fact that there are no temporal analogs to octave and enharmonic equivalence and that there are no tonal analogs to various limits on our temporal perception and acuity. These non-isomorphisms in the spatial representations for pitch and meter call into question the validity of broader claims about the unity of pitch-time relationships.

2. Mapping PC Space

Recent work in neo-Riemannian tonal theory has produced sophisticated representations of tonal space. Starting with the graphic arrays or tonnetze developed by Riemann and Oettingen, a number of theorists have developed spatial representations of pitches, triads, and tonal distance that is neutral with respect to any particular chord progression or key.[2] Tonnetze may thus be viewed as basic substrates to various chord and key relationships. Hyer (1995, 102) reproduces the following figure from Riemann (1914–15):

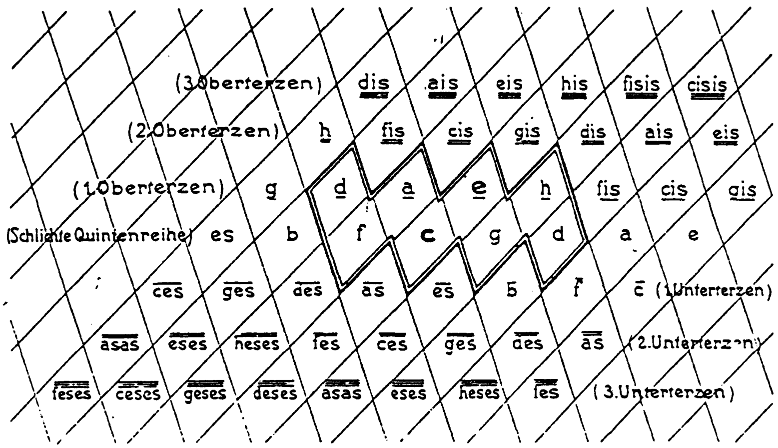

Figure 1. Riemann's Tonnetz (1914–15), from Hyer 1995, figure 1, p. 102

Of this diagram Hyer says, "It is as if, using C as ground zero, Riemann has taken the combinatorial intervals of the *Klang* and strewn them in all imaginable directions, mapping out an abstract terrain of harmonic consonances" (101). Hyer also notes: "A crucial feature of the grid is its extreme chromaticism: fig. 1 represents an unbounded conceptual area containing an infinite number of different pitches, no two of which are identical. Because Riemann assumes just intonation, the lattice is infinitely extensible on all sides" (105). Hyer then re-imagines Riemann's space under the constraints of enharmonic and octave equivalence, and notes that in so doing Riemann's "tabular representation of tonal relations gives rise to remarkable algebraic and topological properties" (106). The result is the following figure (Hyer's fig. 3, 119):

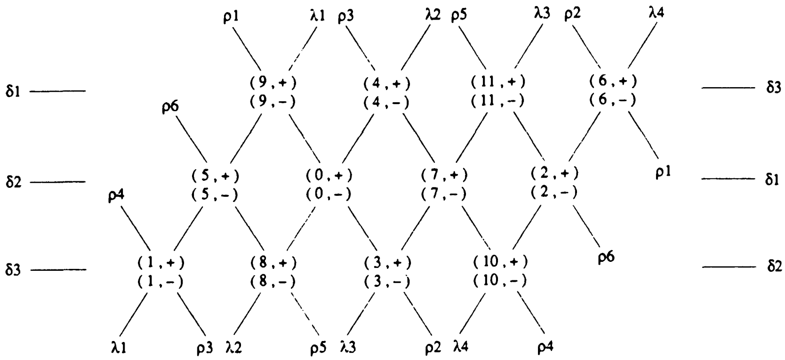

Figure 2. Hyer's Tonnetz of P5, M3 and m3 cycles,
from Hyer 1995, figure 3, p. 119

Hyer notes that when one connects the edges in the manner shown in the diagram, the result is a doughnut or torus. Two dimensions are not adequate for mapping the spatial relationships among pitch classes in a tonnetz; to capture the continuities generated by enharmonic and octave equivalence requires at least three.

More recently Cohn (1997, 10) has produced a generalized form of the tonnetz and then has noted the special properties of the triadic tonnetz in the context of 12-tone equal temperament. Cohn's abstract tonnetz is given in figure 3.

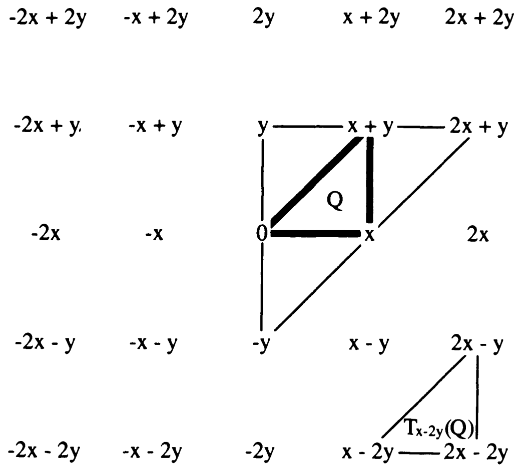

Figure 3. Cohn's Abstract Tonnetz, from Cohn 1997, figure 6, p. 10

Like Hyer, Cohn notes that "if x and y [Cohn's variables for horizontal and vertical relationships in the tonnetz] are assigned to acoustically pure intervals (as in Euler, etc.), or to intervals in pitch-space, then the structure implicitly projects into an infinite plane. The realizations [of the tonnetz] that will hold our focus are generated by equally tempered intervals in some modular system, where the modular congruence represents octave equivalence. In such interpretations, both x and y axes become cyclic rather than linear, and the plane . . . therefore projects into itself as a torus" (Cohn 1997, 11–12).

Music theorists are not alone in recognizing the toroidal shape of tonal space. Researchers in music perception and cognition have empirically measured the goodness-of-fit for notes and chords in a tonally primed context, and they too have mapped tonal space onto the surface of a torus. Krumhansl and Kessler (1982) give the following representation of interkey distances:

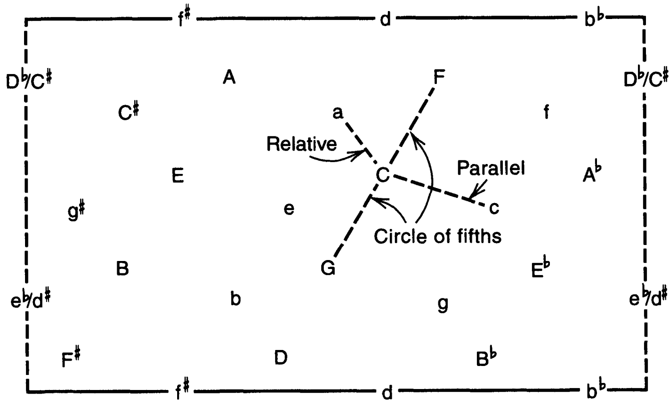

Figure 4. Krumhansl and Kessler's (1982) map of interkey distances, from Lerdahl 1988, figure 26, p. 342.

Nor were they the first to note the three-dimensional nature of tonal space. The cognitive psychologist Longuet-Higgins (1962, 280) lays out his own version of a two-dimensional tonnetz and also notes that the musical space is "three dimensional if one gives due respect to the octave" (i.e., noting how his mapping will wrap into a cylinder).[3]

Lerdahl (1988), strongly influenced by Krumhansl and Kessler, as well as by the work of Deutsch and Feroe, has produced a somewhat different mapping of tonal space. Lerdahl notes: "There are two general problems with current tonal pitch spaces: they are too symmetrical and they address only one level of pitch description" (Lerdahl 1988, 317). For as can be seen in figure 4, there is not a constant spacing between adjacent keys, but varying degrees of relative proximity. This is in part because Krumhansl and Kessler have not given a generic map of tonal space, but rather one that is related to and generated by a particular tonic. Lerdahl also takes this approach, as he develops a mapping of tonal distance that is sensitive to differences among structural levels, treating pitch proximity, triadic proximity, and tonal proximity as separate but interrelated distance metrics. Ultimately, however, Lerdahl produces a mapping of chord space which is symmetrical and which is toroidal:

vii°	ii	IV	vi	I	iii	V
iii	V	vii°	ii	IV	vi	I
vi	I	iii	V	vii°	ii	IV
ii	IV	vi	I	iii	V	vii°
V	vii°	ii	IV	vi	I	iii
I	iii	V	vii°	ii	IV	vi
IV	vi	I	iii	V	vii°	ii

Figure 5. After Lerdahl's representation of chordal space (Lerdahl 1988, figure 13, p. 326; see also Lerdahl 2001, figure 2.14, p. 57)

Lerdahl then develops a somewhat different mapping for harmonic regions. He then combines them, and notes that "at this point, the geometry becomes hard to visualize [as it is a hypertorus] . . . so I will not attempt it" (Lerdahl 1988, 332).

Leaving Lerdahl's hyper-torus aside for the moment, we can give the following summary description of Tonnetz representations of tonal space:

- In a tonnetz there is a uniformity of relationships from one location to another—all of the links form the same pattern, generating a uniform space.
- Topologically no location in the tonnetz is privileged, though in applying the tonnetz to real musical surfaces, a particular location assumes the role of "tonic" such that other locations are heard relative to its location (at least as long as that particular tonic holds sway).[4]
- Given enharmonic and octave equivalence, the tonnetz requires at least 3 dimensions (i.e., a toroidal surface) for a representation which preserves all of the relationships between adjacent tonal elements.

3. A Graph Theory Interlude.

A tonnetz is a kind of graph, a finite set of one or more *vertices* connected (or not) to each other by a set of zero or more *edges*. Here are a number of graphs:

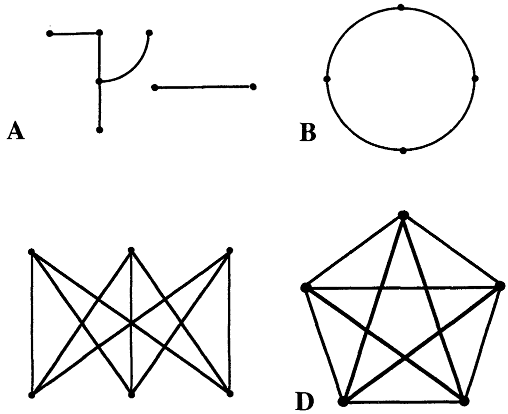

Figure 6. Some basic graph types:
A. Simple unconnected graph;
B. A cyclical graph;
C. The bipartite graph K3, 3, also known as the utility graph;
D. The complete graph K5

Figure 6A is a graph, although not a continuous one. It consists of seven vertices and four edges. We can specify edge-vertex relations by giving the *degree* of each vertex in a graph, that is, the number of edges that meet at each vertex. So in figure 6A there are five vertices of degree 1, one vertex of degree 2, and one vertex of degree 3.

Figure 6B is a cyclical graph. In a cycle one has the same number of edges and vertices, and each vertex is of degree 2. Figure 6C is a bipartite graph K3, 3 (K indicates completeness, and "3, 3" indicates the number of elements in each half of the bijection), as it shows how one set of elements relates to another—in this case, each of the three vertices on top connect to all three vertices on the bottom. It is also known as the utility graph, as it illustrates a classic problem of whether three utility companies can hook up their services to three houses without crossing their pipes or wires (they cannot, as we shall see). Finally, figure 6D is a *complete graph*, here K5. A complete graph is defined as the graph of some set of vertices in which each vertex is connected to every other. Thus for a complete graph consisting of N vertices, each vertex will be of degree N-1.

Two graphs are isomorphic if they share the following properties:

(a) they have the same number of vertices,
(b) they have the same number of edges,
(c) the same distribution of degrees (for each vertex in each graph)
(d) the same number of "pieces" in each graph

The first two requirements should be obvious, since two isomorphic structures each need to be comprised of the same number of elements. The last two insure that those elements stand in the same relationship(s) to each other. It does not matter whether the edges are straight or curved, nor does the relative spacing of the vertices matter. Here are some more examples:

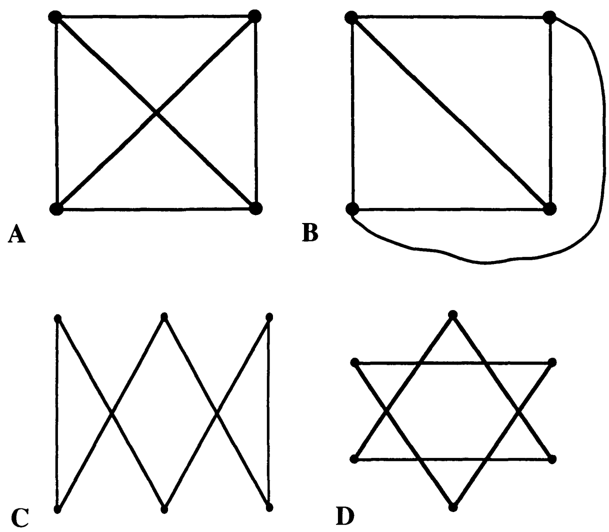

Figure 7. Isomorphic and non-isomorphic graphs:
A. Complete Graph K4;
B. Also K4;
C. Connected graph containing 6 vertices, all of degree 2;
D. Non-connected graph containing 6 vertices, all of degree 2

Figure 7A is the complete graph K4—a set of four vertices, each connected to each other. Figure 7B is also K4, with one edge drawn "outside the box." Trudeau makes the following suggestion in regards to envisioning isomorphisms:

> Think of a graph as a network of steel balls (vertices) and rubber bands (edges). We assume that the balls will remain in whatever position we place them and that the rubber bands will never break. Under this interpretation, isomorphic graphs are graphs that can be arranged to look like one another."[5]

One other thing to note: edges can "pass through each other" when you are reconfiguring a graph (if we stipulated that they could not, we would be talking about knot theory). Therefore, we simply lift the left-upward diagonal in figure 7A, and stretch it outside of the box to create 7B. Both 7A and 7B have the same number of vertices, the same number of edges, all vertices are of the same degree, and both are in one piece. 7A and 7B are isomorphic. In Figures 7C and 7D we have a pair of graphs that have the same number of vertices, the same number of edges, and all vertices are of the same degree, but they are not isomorphic. Notice that 7C is in one piece—it is possible to "walk" from one vertex to any other vertex via edge connections, while 7D is in two pieces—if you are on the upper triangle, you can walk to two other vertices, but you cannot follow an edge to the lower triangle. Another way to put it is that 7D is a graph that contains two *subgraphs*, here each a 3-cycle, but there are no common edges between each subgraph.

Figures 7A and 7B (that is, two versions of K4) bring up another important aspect of graph theory, namely *planarity*. A graph is planar if it is possible to draw it in a two-dimensional plane without edge-crossings. Because we can draw K4 without crossing the diagonals, it is planar. K3, 3 and K5 are non-planar.[6] If one graph is planar, and another is non-planar, they cannot be isomorphic (though subgraphs of each can be).

Planar graphs give particular definitions to the plane they inhabit. Here is a definition from Trudeau: "When a planar graph is actually drawn in a place without edge crossings, it cuts the plane into regions called *faces* of the graph" (Trudeau 1993, 99). The plane itself is counted as one face. Thus the planar drawing of K4 (figure 7B) cuts the plane into four faces—two triangles, the irregular space between the exterior edge and the south and east sides of the square, and the plane itself.

We have already noted the difference between 7C and 7D. Here is a more formal definition of that difference, modeled after Trudeau (p. 97): A *walk* in a graph is a sequence $A_1, A_2, A_3 \ldots A_n$ of not necessarily distinct vertices in which A_1 is joined by an edge to A_2, A_2 by an edge to A_3, and so on, through A_n. If every pair of vertices in a graph is joined by a walk, then that graph is said to be *connected*. Thus in 7D, there is no walk which joins opposite points of the star.

When we combine planarity and connectivity, the result is another means of characterizing graphs and graphic similarities: "A graph is *polygonal* if it is planar, connected, and has the property that every edge borders on two different faces" (Trudeau 1993, 100). Polygonal graphs form regular or irregular polygons in the two-dimensional plane. All of the graphs in figure 7 are polygonal. The graph in figure 6A is not polygonal, both because it is not connected and because all of its edges border on the same face (i.e., they are all "surrounded" by the plane itself). Most tree diagrams are not polygonal for this latter reason. If figures 6C and 6D were planar they would be polygonal, as they are connected and every edge borders on two different faces.

We have now gone, though rather quickly, through enough graph theory to give a fairly thorough characterization of various tonnetz representations. Riemann's original conception of the tonnetz is almost polygonal: it is connected, every edge borders on at least two faces, and it is planar. The reason we must qualify it as "almost" polygonal is that by definition a graph is comprised of a finite set of vertices (and hence edges), whereas one can extend the Riemannian tonnetz ad infinitum. When we include octave and enharmonic equivalence, the graph becomes non-planar.

Here is a way to get a grasp of the non-planar aspects of the tonal torus. Look again at figure 3, Cohn's "parsimonious tonnetz." The vertical axis is for major thirds, the horizontal for minor thirds. If we wrap these axes to capture PC equivalence, the major-third cycles can be thought of forming "loops" which define the cross-section of a cylinder, and the minor-third cycles link the four major-third cycles, joining the ends of the cylinder to form the torus. We can then add the fifth cycle—the diagonal—as a line (more precisely, a connected series of edges) that makes three interleaved spirals around the torus, ending on the point where it started. Following the "fifth spiral" around the torus takes once through each and every vertex just once—this is known as a "Hamilton walk" through the vertices of a graph. Better known is the "Euler walk" through a connected graph: a path which goes through each and every edge just once. Because the tonal torus is a connected *regular* graph, in that every vertex is of the same degree, and since that degree is even (in our case, 6, since every vertex connects two major thirds, two minor thirds, and two perfect fifths), the tonal torus also has an Euler walk.

As was noted above, Lerdahl takes an alternative approach to his representation of tonal space, one which privileges diatonic relationships. We can show that even without taking chromatic additions to the diatonic set into account, the result is non-planar. Let us start with a major diatonic collection, using C as a tonal generator. Figure 8 represents all of the possible intervallic connections between those seven PCs in the form of the complete graph K7:

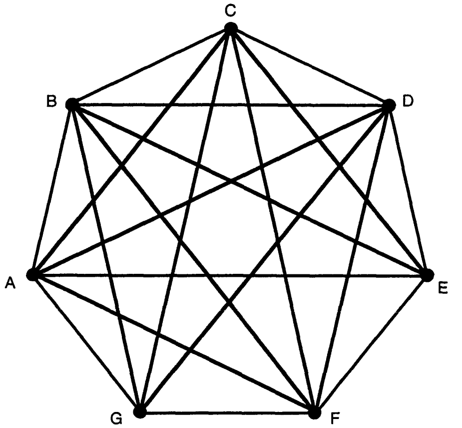

Figure 8. K7 representation of diatonic interval set

Figure 8 contains diatonic adjacencies (the perimeter of the heptagon), thirds, and fifths. It thus contains all of the diatonic "alphabets" used by Lerdahl to measure tonal distances relative to a given tonic (1988, 322–27). The graph in figure 8 is connected, regular (by definition, in K7 every vertex is of degree 6), and non-planar. Its non-planarity can be proven in the following manner. Every complete graph of Kn also contains as subgraphs of the complete graphs K(n–1), K(n–2) . . . through K1 (K1 is the graph which consists of but a single vertex). Since every complete graph Kn consists of n vertices and $\frac{n(n-1)}{2}$ edges, and (by definition) each vertex is connected to every other, then if one erases one vertex from Kn, and the attendant edges that connect the (now missing) vertex to the other vertices, one has a graph with: $\frac{n(n-1)}{2} - (n-1) = \frac{n(n-1) - 2(n-1)}{2} = \frac{n^2 - 3n + 2}{2} = \frac{(n-2)(n-1)}{2}$ edges and *n*–1 vertices, i.e., a complete graph K(n–1). Thus K7 contains K6, and mutatis mutandis, K6 contains K5. Therefore K7 contains K5. K5 has been shown to be non-planar. Therefore K7 is also non-planar; it in fact also leads to "a pleasant toroidal embedding" (West 1996, 281), given in figure 9A. Notice here that the four corners of the planar representation of the K7 correspond to a single point on the surface of the torus. Notice also that figure 9A has obvious similarities to the structure of the tonnetz, stemming from the fact that in both graphs each vertex is of degree 6. Figure 9B is Lerdahl's toroidal representation of chordal space, a reconfiguration of figure 5 (Lerdahl 2001 figure 2.15,

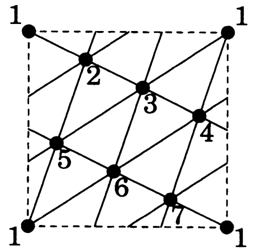

Figure 9A. K7 representation on the surface of a torus, from West 1996, p. 281

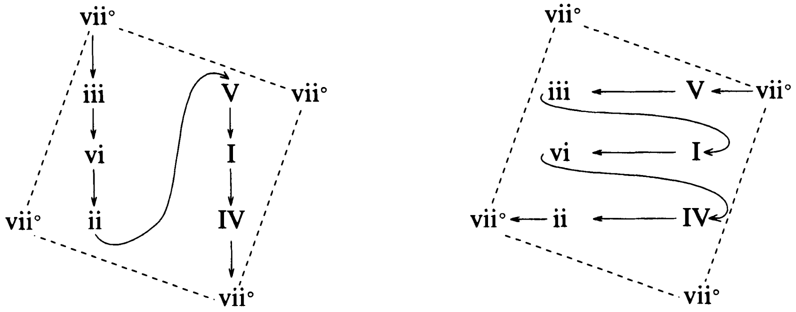

Figure 9B. Toroidal representation of chordal space, from Lerdahl 2001, fig. 2.15, p. 58

58), that indicates various traversals from chord to chord. Its similarities with the toroidal representation of K7 are obvious.

4. Mapping Metric Space

As stated earlier, the tonnetz forms a substrate to various triadic and tonal relationships, and as such is an excellent representation of the "space" in which chords and keys "move." What would be the best temporal analog to the toroidal tonnetz? I would posit that it should be a representation of metric relationships, as a meter is a similar temporal substrate for our experience and understanding of musical time, and meter often gives temporal definition to durational patterns and relationships. Just as tonal systems (i.e., twelve-tone equal temperament) and scales define and constrain pitch-to-pitch intervals and larger melodic constellations, so too do metrical systems define and constrain particular rhythmic motives and their possible arrangements. Thus we seek to graph a meta-metric system which serves as the substrate to a variety of possible meters (and hence durational relationships).

An obvious way to start would be to try and construct a temporal analog of a Riemannian tonnetz, a "zeitnetz," as in figure 10:

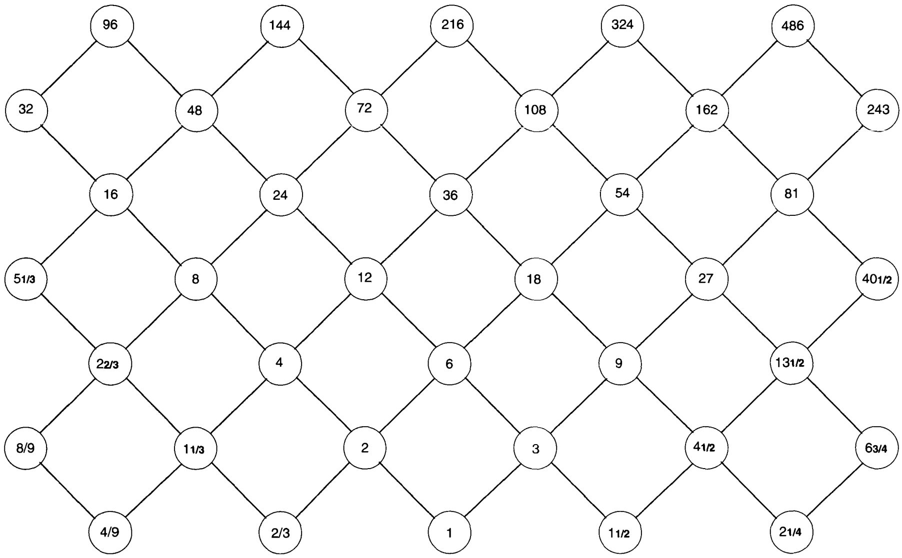

Figure 10. Zeitnetz representation of duple and triple metric relationships

Figure 10 is a partial representation of such a network, in which each vertex represents a particular periodicity, and each connects to four other vertices. So, for example, the vertex labeled "12" connects to two larger periodicities (24 and 36) as well as two smaller periodicities (4 and 6). Duple relationships are mapped on the diagonals that extend downward from left to right, while triple relationships are mapped on the diagonals that extend upward from left to right. This zeitnetz recursively maps the basic metric relationships of binary versus ternary beat orderings (duple versus triple measures) and binary versus ternary beat subdivisions (simple versus compound time). This network is simply the set of all values of form $2^m \times 3^n$ where m and n are integers.

Like mappings of tonal space, this matrix is uniform—the relationship from any vertex to its adjacent vertices is constant (each vertex is of degree 4). As such, all levels of structure are "metrically alike" in this matrix, and while I would imagine that Stockhausen might find this attractive, I would argue that there are important distinctions between metric levels that a mapping of the metric space should capture. Certain levels—the measure and especially the tactus—are more essential than others. If a layer of subdivision or hypermeter should drop out, we still have metric and temporal continuity, whereas if the beat disappears, there is a palpable lack of motion. Thus the beat or tactus serves as the fundamental substrate for any metric system.[7]

Thus the uniformity of figure 10 is a problem, and hence my first attempt at mapping metric space, what I shall term M-space: a conceptual space that lays out the hierarchic relationships among the most common meters in western tonal music:[8]

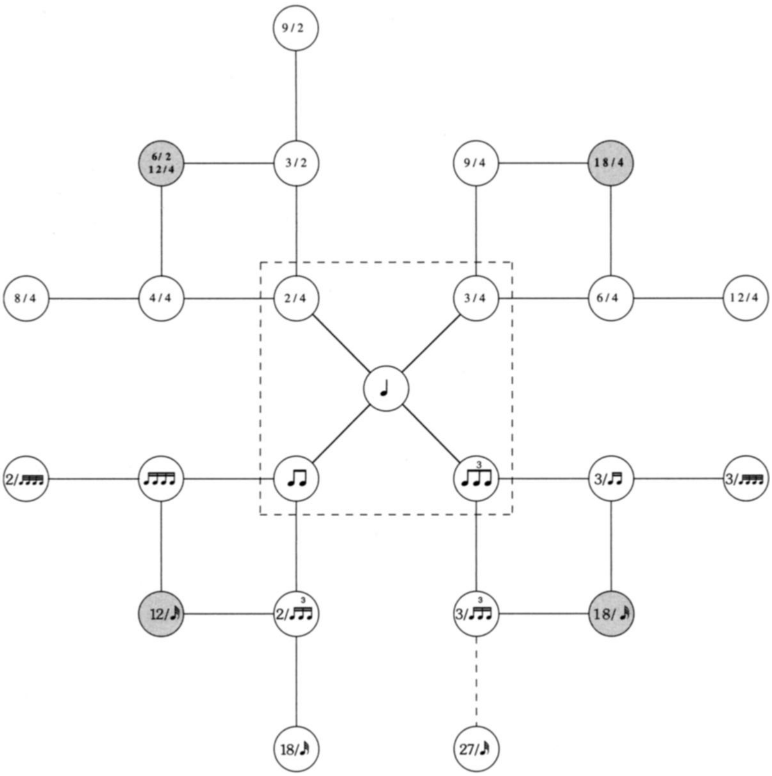

Figure 11. Graph (rooted tree) of M-space

As in figure 10, we have a configuration of vertices and edges. Each vertex in M-space indicates the organization of a given level of a musical time, with the beat level (for convenience, represented by a quarter-note) serving as the origin for the space. This reflects the centrality of the beat level. Each vertex in this graph represents a level of periodic articulations that is linked to higher and lower levels in terms of either 2:1 or 3:1 ratios. The horizontal edges represent binary relationships, while the vertical edges represent ternary relationships. The four basic metric types (duple versus triple and simple versus compound) are enclosed in the dashed box. Vertices that involve concatenations of beats and larger units are measures and hypermeasures (the upper two quadrants), and vertices that involve fractions of the central beat are subdivisions (the lower two quadrants). You will notice that the upper two quadrants contain most of standard time signatures in western music.

In terms of mathematical graph theory, we have a rooted tree whose central vertex is of degree four, with all other vertices are (at least in principle) of degree three—the gray vertices are used to indicate overlapping hierarchic configurations (for example, 6/2 vs. 12/4) which have identical periodicities, a bit of fudging in order to preserve the "vertical = ternary" and "horizontal = binary" relationships for the edges. Nonetheless one can draw this tree without edge crossings, and hence it is planar.

This representation of M-space also emphasizes the sharp perceptual differences between duple and triple meters and binary versus ternary subdivisions. Clarke (2000) summarizes the phenomenon that he and others have studied:

> The data . . . showed the characteristic features of categorical perception (a disjunction in the identification function as subjects switch from one perceptual category to another, coupled with a peak of discriminability when pairs of stimuli are taken from either side of the category boundary), together with a metrical effect causing the category boundary to shift so as to make a larger proportion of the stimulus continuum consistent with the prevailing metre. In simple terms, the effect of the metrical context is to cause subjects to perceive potentially ambiguous rhythms in a fashion which supports and confirms the prevailing metre.[9]

Thus the separation between the duple and triple branches of figure 11 reflects how we construe metric systems in terms of perceptual categories.

How many different metric configurations are represented figure 11? First, we must give a minimal definition for a single meter. After Yeston 1976 I will stipulate that a meter requires at least two coordinated periodicities. I will further stipulate that at least one of those levels must be the central beat. Meters may vary from very thin (a two-tiered duple or triple ordering of beats) to very thick (e.g., 6/2 comprised of running 16th sextuplets!). Thus the answer to the question posed above is 195, as it is the product of all of the measure vertices (including the central beat) times all of the subdivision vertices (also including the central beat level), minus one (to eliminate mapping the central beat onto itself). Though this question may seem trivial, as it turns out it is not. While one could add additional edges and vertices to figure 11, the limitations of human temporal perception serve as a significant constraint on the extent of M-space. Thus while formally we may imagine an M-space that is topologically open, extending ad-infinitum, we are able to perceive only a small region of its extended terrain. Just as our perception of octave equivalence determines the shape of PC space, so too do our perceptual and cognitive capacities influence the shape and extent of M-space.

In his influential textbook, Peter Westergaard (1975, 274) gave a "chart of useful tempos," given as figure 12.

Number of beats/minute	IOI between beats	Comment
30	2 seconds	too slow to be useful
42	1.414 seconds	very slow
60	1 second	moderately slow
84	700 milliseconds	moderate
120	500 milliseconds	moderately fast
168	350 milliseconds	very fast
240	250 milliseconds	too fast to be useful

Figure 12 (After Westergaard 1975, p. 274)

To Westergaard's chart I have added timings in seconds and milliseconds. Since Westergaard mentions only beats, one must make some extrapolations about the timing values of higher and lower metric levels. Nonetheless, his comments accord nicely with the findings of music psychologists: one cannot metrically track (that is to say, count) temporal events that are faster than about 80–100 milliseconds (Westergaard's "too fast to be useful" tempo) and 2 seconds is about the outer limit for temporal continuity.[10] It should thus be obvious that tempo changes—chances in the duration of the central beat—will have an effect on the scope and effect of M-space. When we give a specific tempo value to the central beat rate, we may now speak of Meter/Tempo space (M/T space). Graphing M/T-space allows us to see the systematic interaction(s) between meter, tempo, and various perceptual and cognitive thresholds.

Figure 13 shows the various timing relations between vertices at a tempo of 92 beats/minute (a 650ms duration). As one can see, the longest periodicity is 12 seconds, while the shortest is 24 milliseconds—and both

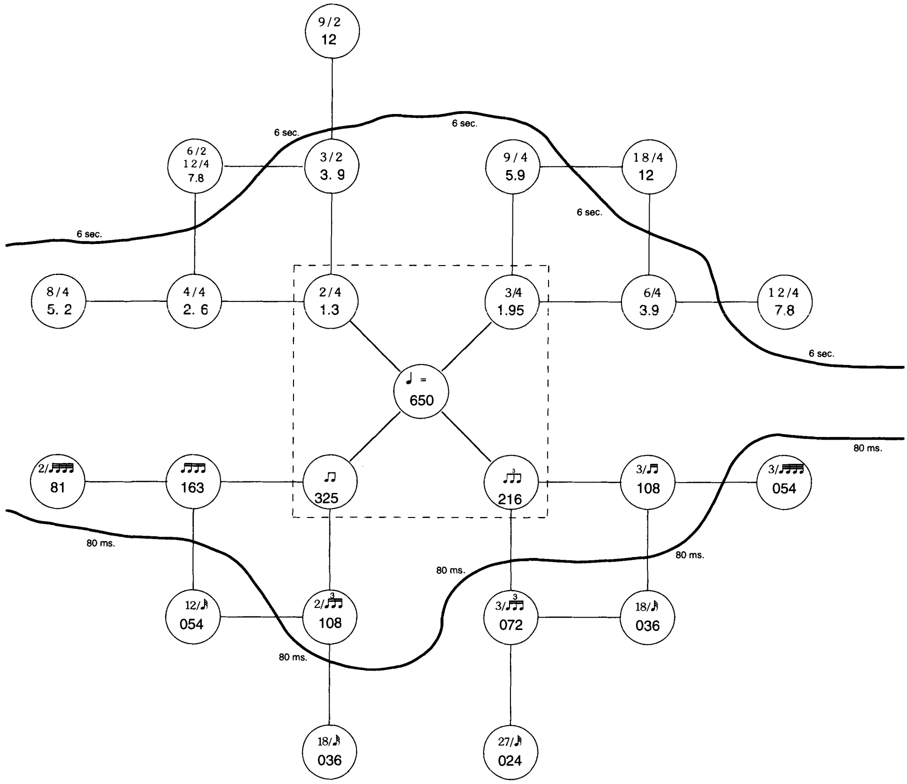

Figure 13. Graph of M/T space w/650ms beat duration

of these lie outside the "metric envelope" of our perceptual and cognitive capacities. On this example I have written temporal contour lines that mark the edges of metric envelope. First, note also that some nodes lie near the edge of the envelope (81ms and 5.9 sec). While it may be possible to hear these metric relationships, it may require more attentional effort as they fall near a perceptual or cognitive limit. As one also might expect, the number of possible configurations in this tree is skewed toward 2:1 ratios. Note also that at this tempo full measures of both duple or triple time fall under the two-second threshold, suggesting that downbeats in either meter will seem strongly connected, and inviting higher levels of metric structure.[11] The metric envelope also truncates various branches of the graph, so that we now have a graph with a central vertex of degree four, intermediate vertices of degree three, and terminal vertices of degree one.

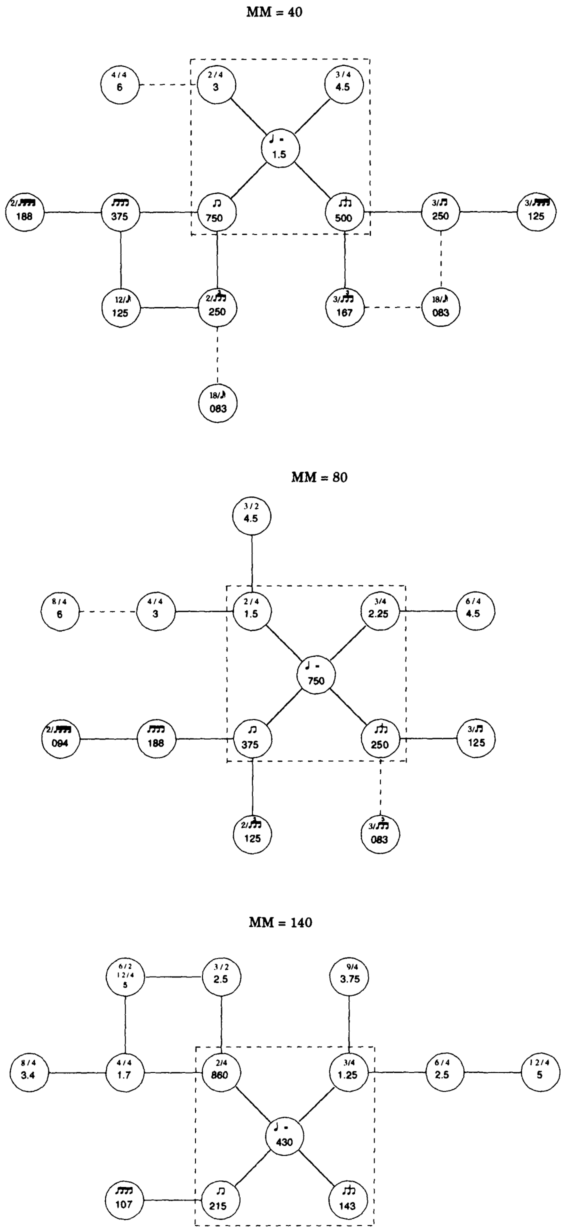

Figure 14. Graphs of M/T space at various tempos

Figure 14 shows three graphs of M/T space at various tempos (vertices outside the metric envelope have been omitted for clarity). As one would expect, at slower tempos there are limits to the number of levels above the beat (as they simply become too long), while, conversely, at faster tempos there are limits on the extent of metric subdivision (as the subdivisions simply get too short). Each graph is yoked to a beat rate that is typical of a particular tempo, and so the three graphs illustrate the metric differences between these six distinct tempo categories.

What is perhaps most interesting is that the number of possible meters in each graph of M/T space is not constant, nor is there a simple linear relation between the number of possible meters in any graph of M/T space and tempo:

Beat Rate (MM/ms)	# of vertices within the metric envelope	MxSD-1	Periodicities in the 600-700ms range?
40 (1500)	12	3x10-1 = 29	N
50 (1200)	13	4x10-1 = 39	Y
60 (1000)	11	4x8-1 = 31	N
72 (833)	11	6x6-1 = 35	N
80 (750)	11	6x6-1 = 35	N
86 (700)	11	6x6-1 = 35	Y
92 (650)	10	5x6-1 = 29	Y
100 (600)	12	7x6-1 = 41	Y
108 (555)	11	8x4-1 = 31	N
120 (500)	11	8x4-1 = 31	N
140 (428)	13	10x4-1 = 39	N
160 (375)	13	10x4-1 = 39	N
180 (333)	12	10x3-1 = 29	Y
200 (300)	12	10x3-1 = 29	Y

Figure 15. Table of the number of nodes in M/T space relative to various tempos

The left-hand column gives the beat rate in the musically familiar term of beats per minute. The next column lists number of vertices present, and this varies only between 13–15. In order to see how tempo changes really effect the extent of M/T space, one must take note of the multiplicative relationships between vertices above versus below the central beat, as indicated in the "M times SD minus ONE" column. Note here the wider variation, from 29 to 41 metric vertices at various tempos. The last column tracks another interesting aspect of tempo change: at some tempos, there are no periodicities present in the range of maximal pulse salience, approximately 600–700ms (see Parncutt 1994). If this periodicity is as important as psychologists and historical sources have suggested, especially in relation to the kinesthetic and somatic aspects of meter, then metric hierarchies which lack a "resonance" in the 600–700ms range will have distinct perceptual and hence musical qualities. An examination of M/T space thus reveals that there may be systemic reasons for our categorical preferences for some tempos and not others.

Small changes in tempo can have a large effect on the number of vertices in M/T space:

Beat Rate (MM/ms)	# of vertices within the metric envelope	MxSD-1	Periodicities in the 600-700ms range?
86 (700)	11	6x6-1 = 35	Y
88 (682)	11	6x6-1 = 35	Y
90 (667)	11	6x6-1 = 35	Y
92 (650)	11	6x6-1 = 35	Y
96 (625)	12	7x6-1 = 41	Y
100 (600)	12	7x6-1 = 41	Y
108 (555)	11	8x4-1 = 31	N
120 (500)	11	8x4-1 = 31	N

Figure 16. Table which tracks changes in the number of nodes in M/T space as tempo shifts from 86–120 beats/minute

This is an overall tempo change of less than 20%—and in this range a tempo change of less than 7% is often not even noticeable (around 7% is the "just noticeable difference" for durational changes in the 250ms to 2 second range). Yet due to the multiplicative and divisive properties of the connections between vertices, these small changes are magnified in terms of the extent of M/T space.

5. Conclusion: M/T Space and Various Tonnetze are Non-Isomorphic

It is hoped at this point that the principal argument of this paper should be fairly obvious to the reader: metric space is planar, tonal space is non-planar; therefore the two spaces are non-isomorphic. And if the two spaces are non-isomorphic, then there are fundamental problems in trying to map elements or relationships (i.e., functions which employ those elements) from one space to another.

Differences beyond planarity versus non-planarity may also be discerned: the tonnetz is regular, with all vertices of the same degree; the graph of M/T space involves vertices of different degrees. The many vertices of degree 1 in M/T space create a large number of "dead ends" in the graph, and so it is not possible to have either an Euler or a Hamilton walk through M/T space. M/T space is a rooted tree, and even if we acknowledge Lerdahl's concerns regarding the role of a tonic in generating a tonal space, the root of a rooted tree is not the same as a tonic in tonal space. That one vertex serves as the origin of a tonnetz does not change the uniformity of the structure of the network itself (and indeed, various marvelous effects of chromatic harmony depend on this to move smoothly to "distant" chords and keys).

Finally, changing the tempo changes both the shape and extent of M/T space. Changing one's initial tonic does not alter the number of vertices or edges in the graph of tonal space—wherever you start, there are always the same number of tonal possibilities, the same number and kind of pathways to other chords or pitch-class complexes (i.e., all edge-relationships remain constant). Similarly, changing one's initial tonic does not change the degree of any other vertex in the tonnetz (i.e., all vertex relationships remain constant). In contrast, changing tempos *does* change the number of edges and vertices in the graph of tonal space, as well as the relationships among them, for example, as higher levels of subdivision that are vertices of degree 3 at slower tempos lose edges (and hence change degree) as the tempo increases.

From the outset Hyer reminds us that tonnetze—both his and Riemann's—are tonal *representations*, in that they are translations from that which is heard and remembered to that which is seen. Hyer notes that "Riemann himself calls our attention to the radical consequences of theorizing hearing in terms of seeing. After commenting on our inclination to imagine low frequencies as dark tones and high frequencies as bright tones, he further observes that 'the hearing of changes in [pitch] is transformed into a vision of changes in location," concluding that musical cognition assumes an '*ultimate identification of the essences of visual and aural imagination*'" (Hyer 1995, 104).[12] This rings true; my account of the differences between various representations of tonality and meter stands or falls on the extent to which they truthfully represent relationships between pitch and time. But the use of representations—whether in musical notation, pictures, or words—is unavoidable in music theory and analysis. What is gained in this exercise is that by trying to follow the same "rules" in constructing graphic representations of tonal and metric relationships, we are forced to confront the differences between them. We also are reminded of how the topologies of both M/T space and PC space—the metric tree and the tonal tonnetz—arise from the combination of formal relationships among their component elements as well as the way human beings hear and understand those relationships: octave and enharmonic equivalence play a strong role in shaping the tonal torus, as do categorical perception and the upper and lower limits of temporal discrimination in shaping and pruning the metric tree. These graphs thus represent not just musical systems, but also our musical psychology. As in all of our musical representations, what we can hear and what we can imagine are intertwined and interdependent.

NOTES

1. To be sure, Lewin is careful to note the different group-theoretic properties of each domain—see chapter 4 and passim.
2. For an overview of recent work in neo-Riemannian theory see Cohn 1998, and indeed all of volume 42.2 of this journal.
3. Longuet-Higgins also somewhat crankily complains that "musical theorists should be, apparently, so ignorant of the two-dimensional nature of harmonic relationships" (248); though he seems to acknowledge Schoenberg's mappings of tonal regions, he was apparently unaware of the work Riemann, Oettingen, and others.
4. Here is where Lerdahl's representation differs most strongly from other neo-Riemannian representations. For Lerdahl, a tonic location is privileged, as it serves to generate the space. In an alternative to this approach, as Hyer has aptly put it, a tonic introduces a kind of "gravitational distortion" into a tonal space, which might account for the differing degrees of tonal distance found by Krumhansl and Kessler, but which does not require a non-uniform structure of the space itself (Hyer 1995, 109).
5. Trudeau's work is an excellent introduction to the basic conceptions and problems of graph theory.
6. To get an intuitive notion of their non-planarity, try and stretch the interior diagonals of K5 outside the pentagram to create a graph without edge crossings, as was done with K4 (hint: leave two diagonals inside).
7. Music psychologists have also recognized the primacy of the beat level of the metric hierarchy: Jones (1992) and Jones and Boltz (1989) speak of a "referent level" which anchors the metric hierarchy; Miller, Scarborough, and Jones (1992) have shown that beat level oscillators do not require as much reinforcement as do higher and lower levels of metric structure; and Fraisse (1987) and Parncutt (1994) have shown that the perception of a beat or pulse is correlated with preferred tempo and subjective rhythmicization.
8. An extended discussion of this representation of metric space, and its relationship to various perceptual and cognitive limits, is given in London 2002.
9. See also Gabrielsson, Bengtsson, and Gabrielsson 1983; Clarke and Windsor 1992; and Windsor 1993.
10. For studies of the 80–100ms threshold see Hirsh, et. al. 1990 and Roeder 1995; for studies of the two-second upper bound for temporal continuity among successive stimuli, see Fraisse 1982; for the limit of the psychological present and its effect on meter see Brower 1993 and Berz 1995.
11. Interestingly, while the duple subdivision is longer than 250ms, the triplet subdivisions are shorter—suggesting that at this particular tempo, there may be categorical differences between simple versus compound subdivision, as 250ms is another important cognitive threshold.
12. On the other hand, cross-domain mappings between auditory and visual phenomena, and even shared forms of mental processing and representation, have been the subject of much research in auditory perception and cognition; the locus classicus for recent work in this area is Bregman 1990; see also Saslaw 1996 and Zbikowski 1998.

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